Problem 1. Let $ABC$ be a triangle with $AB < AC$. Let $\omega$ be a circle passing through $B, C$ and assume that $A$ is inside $\omega$. Suppose $X, Y$ lie on $\omega$ such that $\angle BXA = \angle AYC$. Suppose also that $X$ and $C$ lie on opposite sides of the line $AB$ and that $Y$ and $B$ lie on opposite sides of the line $AC$.

Show that, as $X, Y$ vary on $\omega$, the line $XY$ passes through a fixed point.

Problem 2. Find all functions $f : (0, +\infty) \rightarrow (0, +\infty)$ such that
\[ f(x + f(x) + f(y)) = 2f(x) + y \]
holds for all $x, y \in (0, +\infty)$.

Problem 3. Let $a, b$ and $c$ be positive integers satisfying the equation
\[ (a, b) + [a, b] = 2021^c. \]

If $|a - b|$ is a prime number, prove that the number $(a + b)^2 + 4$ is composite.

Here, $(a, b)$ denotes the greatest common divisor of $a$ and $b$, and $[a, b]$ denotes the least common multiple of $a$ and $b$.

Problem 4. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel performs exactly one of the following moves:
(a) He clears every piece of rubbish from a single pile.
(b) He clears one piece of rubbish from each pile.

However, every evening, a demon sneaks into the warehouse and performs exactly one of the following moves:
(a) He adds one piece of rubbish to each non-empty pile.
(b) He creates a new pile with one piece of rubbish.

What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

Time: 4 hours and 30 minutes

Each problem is worth 10 points