BMO 2021 CYPRUS
38th Balkan Mathematical Olympiad

Language: **English**

Wednesday, September 8, 2021

Problem 1. Let ABC be a triangle with $AB < AC$. Let ω be a circle passing through B, C and assume that A is inside ω. Suppose X, Y lie on ω such that $\angle BXA = \angle AYC$. Suppose also that X and C lie on opposite sides of the line AB and that Y and B lie on opposite sides of the line AC.

Show that, as X, Y vary on ω, the line XY passes through a fixed point.

Problem 2. Find all functions $f : (0, +\infty) \to (0, +\infty)$ such that

$$f(x + f(x) + f(y)) = 2f(x) + y$$

holds for all $x, y \in (0, +\infty)$.

Problem 3. Let a, b and c be positive integers satisfying the equation

$$(a, b) + [a, b] = 2021^c .$$

If $|a - b|$ is a prime number, prove that the number $(a + b)^2 + 4$ is composite.

Here, (a, b) denotes the greatest common divisor of a and b, and $[a, b]$ denotes the least common multiple of a and b.

Problem 4. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel performs exactly one of the following moves:

(a) He clears every piece of rubbish from a single pile.

(b) He clears one piece of rubbish from each pile.

However, every evening, a demon sneaks into the warehouse and performs exactly one of the following moves:

(a) He adds one piece of rubbish to each non-empty pile.

(b) He creates a new pile with one piece of rubbish.

What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

Time: 4 hours and 30 minutes

Each problem is worth 10 points