Problem 1. Determine all differentiable functions $f : \mathbb{R} \to [0, \infty)$ with continuous derivative such that $f^2(x) \leqslant f'(x)$ for all $x \in \mathbb{R}$.

Solution 1. We will show that the only such function is the trivial one, $f(x) = 0$. Suppose instead that there is some x_0 with $f(x_0) > 0$. As $f'(x) \geqslant f^2(x) \geqslant 0$ for every x, the function f is increasing. Hence, for every $x \geqslant x_0$, we have $f(x) \geqslant f(x_0) > 0$.

The function $g(x) = f'(x)/f^2(x)$ is well-defined, continuous and satisfies $g(x) \geqslant 1$ on $[x_0, \infty)$. Therefore

$$x - x_0 = \int_{x_0}^{x} 1 \,\mathrm{d}t \leqslant \int_{x_0}^{x} \frac{f'(t)}{f^2(t)} \,\mathrm{d}t$$

for every $x \geqslant x_0$.

Since $G(x) = -\dfrac{1}{f(x)}$ is continuous on $[x_0, \infty)$ and continuously differentiable on (x_0, ∞) with $G'(x) = g(x)$ for every $x > x_0$, by the Fundamental Theorem of Calculus we get

$$\int_{x_0}^{x} \frac{f'(t)}{f^2(t)} \,\mathrm{d}t = G(x) - G(x_0) = \frac{1}{f(x_0)} - \frac{1}{f(x)}$$

for every $x \geqslant x_0$. Therefore

$$0 \leqslant \frac{1}{f(x)} \leqslant \frac{1}{f(x_0)} + x_0 - x, \tag{1}$$

for every $x \geqslant x_0$, which is impossible, as the right hand side eventually becomes negative.

Solution 2. We define x_0 in the same way as in Solution 1. Define $h(x) = x + \dfrac{1}{f(x)}$ for $x \geqslant x_0$ and note that it is continuous on $[x_0, \infty)$ and differentiable on (x_0, ∞) with $h'(x) = 1 - f'(x)/f^2(x) \leqslant 0$ for every $x \in (x_0, \infty)$. Therefore h is decreasing which also leads to (1).

Solution 3.

Suppose that there is some x_0 such that $f(x_0) > 0$. We will prove the following lemma.

Lemma. *For every $x \geqslant x_0$ and every non-negative integer n, we have*

$$f(x) \geqslant 2^{n+2} \left(\frac{f(x_0)}{4} \right)^{2^n} (x - x_0)^{2^n - 1}$$

Proof. We have that $f'(x) \geqslant f^2(x) \geqslant 0$, and therefore $f(x)$ is increasing. Thus, we have that, for every $x \geqslant x_0$,

$$f(x) \geqslant f(x_0) = 2^{0+2} \left(\frac{f(x_0)}{4} \right)^{2^0} (x - x_0)^{2^0 - 1},$$

i.e., the Lemma is true for $n = 0$.

Suppose that the Lemma is true for $n = k$. We have that, for every $x \geqslant x_0$,

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(u)\,\mathrm{d}u \geqslant \int_{x_0}^{x} f^2(u)\,\mathrm{d}u.$$

Hence, by applying the induction hypothesis, we deduce that

$$\begin{aligned} f(x) &\geqslant 2^{2k+4} \left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \int_{x_0}^{x} (u - x_0)^{2^{k+1}-2}\,\mathrm{d}u \\ &= 2^{2k+4} \left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \left[\frac{(u - x_0)^{2^{k+1}-1}}{2^{k+1}-1}\right]_{x_0}^{x} \\ &> 2^{2k+4} \left(\frac{f(x_0)}{4}\right)^{2^{k+1}} \frac{(x - x_0)^{2^{k+1}-1}}{2^{k+1}} \\ &= 2^{(k+1)+2} \left(\frac{f(x_0)}{4}\right)^{2^{k+1}} (x - x_0)^{2^{k+1}-1}. \end{aligned}$$

□

Now choose $x = x_0 + \frac{4}{f(x_0)}$. By the Lemma, we have

$$f(x) \geqslant 2^n f(x_0)$$

for every n. Taking $n \to \infty$, we arrive at a contradiction. Hence, f must be identically zero.