Problem 2. Let A and B be $n \times n$ matrices with complex entries for which

$$A^2 + B^2 = 2(AB - BA).$$

Prove that $A^2 + B^2$ is a nilpotent matrix.

An $n \times n$ matrix M is called *nilpotent*, if there is a positive integer k such that $M^k = O_n$.

Solution.

Let $C = AB - BA$. The given equation is $A^2 + B^2 = 2C$. Define

$$L = A - iB, \quad \text{and} \quad M = A + iB$$

Then

$$LM = (A - iB)(A + iB) = A^2 + B^2 + i(AB - BA) = 2C + iC = (2 + i)C.$$

and

$$ML = (A + iB)(A - iB) = A^2 + B^2 - i(AB - BA) = 2C - iC = (2 - i)C.$$

This gives

$$ML = \frac{2-i}{2+i} LM = \frac{(2-i)^2}{5} LM = \frac{3-4i}{5} LM.$$

Let $\alpha = \frac{3-4i}{5}$. Note that $|\alpha| = 1$. So $\alpha = e^{i\vartheta}$ where $\cos\vartheta = \frac{3}{5}$. By Niven's Theorem, since $\cos\vartheta \in \mathbb{Q} \setminus \{0, \pm\frac{1}{2}, \pm 1\}$, then ϑ is not a rational multiple of π. Thus for every positive integer k we have $\alpha^k \neq 1$.

For any $k \geqslant 1$, the relation $ML = \alpha LM$ implies $(ML)^k = \alpha^k (LM)^k$. Taking the trace of both sides and using the cyclic property $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ we get

$$\operatorname{tr}((LM)^k) = \operatorname{tr}((ML)^k) = \alpha^k \operatorname{tr}((LM)^k).$$

Then

$$(1 - \alpha^k) \operatorname{tr}((LM)^k) = 0$$

and since $\alpha^k \neq 1$, then $\operatorname{tr}((LM)^k) = 0$.

Since this holds for every $k \in \{1, 2, \ldots, n\}$, then LM is nilpotent. Since

$$A^2 + B^2 = 2C = \frac{2}{2+i} LM,$$

then $A^2 + B^2$ is also nilpotent.

Solution 2 After arriving at $ML = \alpha LM$, and proving that a is not a root of unity, we can also proceed as follows.

Suppose that LM is not nilpotent. Then, it will have an eigenvalue $\lambda \neq 0$. Let v be a corresponding eigenvector, so that $LMv = \lambda v$. Then, $MLv = \alpha\lambda v$, giving that $\alpha\lambda$ is an eigenvalue for ML. On the other hand, we know that ML has the same spectrum as LM. Hence, $\alpha\lambda$ is an eigenvalue of LM. Inductively, $\alpha^n\lambda$ is an eigenvalue for every n. As $\lambda, \alpha \neq 0$, and α is not a root of unity, this gives us infinitely many eigenvalues, which is a contradiction.

Remark. We note the following two alternative ways to deduce that $\alpha = \frac{3-4i}{5}$ is not a root of unity without using Niven's Theorem:

(a) We have $(5\alpha - 3)^2 = (-4i)^2$, which gives that $p(x) = 5x^2 - 6x + 5$ is the minimal polynomial of α. In order for α to be a root of unity, it must be a root of $x^m - 1$ for some m. Hence, $p(x)$ must divide $x^m - 1$. As $p(x)$ is irreducible and non-monic, while $x^m - 1$ is monic, this contradicts Gauss's Lemma.

(b) Note that $(3-4i)^2 = -7-24i \equiv 3-4i \bmod 5$ and inductively we get $(3-4i)^k \equiv 3-4i$. So $(5\alpha)^k \neq 5^k$.