Problem 3. Let A and B be $n \times n$ matrices with complex entries for which $A^2 = AB - BA$ and $\operatorname{rank}(A - B) = 1$. Prove that $ABA = O_n$.

Solution 1.

Since $AB - BA = A^2$, then $AB - BA$ commutes with A and so (by Jacobson's Lemma) it is nilpotent. Thus A^2 is nilpotent and therefore A is nilpotent.

Assume $\operatorname{Im}(A-B) = \langle x \rangle$. We have $A(A-B) = -BA$ so it is enough to show that $A^2x = 0$.

Let $K = A - B$ and observe that

$$KA - AK = A^2 - BA - (A^2 - AB) = AB - BA = A^2.$$

Since A is nilpotent, there is a minimal k such that $A^k x = 0$. We have

$$A^{k+1} = A^{k-1}A^2 = A^{k-1}(KA - AK) = A^{k-1}KA - A^kK$$

Since $Kx \in \langle x \rangle$ then

$$A^{k-1}KAx = A^{k+1}x - A^kKx = 0$$

Since $KAx \in \langle x \rangle$, and $A^{k-1}(KAx) = 0$, by definition of k we get $KAx = 0$. We have $Kx = \lambda x$ for some $\lambda \in \mathbb{C}$. Thus

$$A^2x = (KA - AK)x = -AKx = -\lambda Ax.$$

So either $Ax = 0$, or Ax is an eigenvector of A which together with nilpotency implies $\lambda = 0$.

In both cases we have $A^2x = 0$.

Solution 2.

Assume $\operatorname{Im}(A-B) = \langle x \rangle$. We have $A(A-B) = -BA$ so it is enough to show that $A^2x = 0$.

Let $Ax = y$ and pick $\lambda, \mu \in \mathbb{C}$ such that $(A - B)x = \lambda x$ and $(A - B)y = \mu x$. We have

$$A(A-B)x = -BAx \implies \lambda y = -By \implies By = -\lambda y.$$

This implies that $Ay = \mu x + By = \mu x - \lambda y$

$$A(A-B)y = -BAy \implies \mu y = -B(\mu x - \lambda y) = -\mu(y - \lambda x) - \lambda^2 y$$

This leads to

$$(\lambda^2 + 2\mu)y = \lambda\mu x.$$

So either $\mu = -\lambda^2/2$, or $y = \alpha x$ for some $\alpha \in \mathbb{C}$.

In the first case we get $-(\lambda^3/2)x = 0$, and since $x \neq 0$, then $\lambda = 0$. But then $Ax = Bx = y$ and so

$$A^2x = (AB - BA)x = (A - B)y = \mu x = 0.$$

In the second case $Ax = y = \alpha x$ and $Bx = y - \lambda x = (\alpha - \lambda)x$ so

$$A^2x = (AB - BA)x = A(\alpha - \lambda)x - B\alpha x = (\alpha - \lambda)\alpha x - \alpha(\alpha - \lambda)x = 0.$$

Solution 3.

As in Solution 1, A is nilpotent. Writing $K = A - B$ we have

$$A^2 = KA - AK$$

and so A^2 is the sum of two rank-1 matrices. Then $\operatorname{rank}(A^2) \leqslant 2$ and since A is nilpotent, then $\operatorname{rank}(A^3) \leqslant 1$ and therefore $A^4 = O_n$.

Note that

$$ABA = A(A - K)A = A^3 - AKA$$

and

$$A^3 = A(KA - AK) = AKA - A^2K.$$

So $ABA = -A^2K$.

Since $\operatorname{rank}(K) = 1$, then $K = uv^*$ for some non-zero vectors $u, v \in \mathbb{C}^n$.

Case 1. Suppose $A^3 = 0$. Then

$$0 = A^4 = A^2(KA - AK) = A^2KA = A^2uv^*A.$$

If $v^*A = 0$, then $KA = 0$, so $A^2 = -AK$ and therefore $-A^2K = A^3 = 0$. Otherwise, $A^2u = 0$ and so again $A^2K = A^2uv^* = 0$.

Case 2. Suppose $A^4 = 0$ but $A^3 \neq 0$ and pick w such that $A^3w \neq 0$. Then

$$0 = A^5w = A^2(A^3w) = (KA - AK)A^3w = -AKA^3w = -Au(v^*A^3w).$$

If $Au = 0$, then $A^2K = 0$ as in Case 1. So we can assume that $v^*A^3w = 0$. Now

$$0 = A^4w = (KA - AK)A^2w = uv^*A^3w - Auv^*A^2w = -Auv^*A^2w.$$

With a similar reasoning as before, we may assume $v^*A^2w = 0$. Now

$$0 \neq A^3w = (KA - AK)Aw = uv^*A^2w - Auv^*Aw = -Auv^*Aw$$

which guarantees that $v^*Aw \neq 0$. But now

$$0 = A^4w = A(A^3w) = -A^2uv^*Aw$$

and since $v^*Aw \neq 0$, then $A^2u = 0$. As in Case 1, we get $A^2K = 0$.

Solution 4.

As in Solution 3, we have $A^4 = 0_n$ and $ABA = -A^2K$. We may assume that $A^2K \neq O_n$ and therefore $\operatorname{rank}(A^2K) \geqslant 1$. We apply Frobenious rank inequality

$$\operatorname{rank}(XY) + \operatorname{rank}(YZ) \leqslant \operatorname{rank}(Y) + \operatorname{rank}(XYZ)$$

to get

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA^3) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA^3).$$

Since $A^2KA^3 = (-ABA)A^3 = -ABA^4 = 0$ and $\operatorname{rank}(A^2K) \geqslant 1$, we get that $\operatorname{rank}(KA^3) = 0$ and therefore $KA^3 = O_n$.

Thus $(A - B)A^3 = O_n$ which leads to $BA^3 = O_n$. Thus $A^2KA^2 = (-ABA)A^2 = O_n$ and so

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA^2) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA^2)$$

gives that $KA^2 = O_n$ and $A^3 = BA^2$. Thus $A^2KA = -ABA^2 = -A^4 = O_n$ and so

$$\operatorname{rank}(A^2K) + \operatorname{rank}(KA) \leqslant \operatorname{rank}(K) + \operatorname{rank}(A^2KA)$$

gives that $KA = O_n$.

From this we get $A^2 = KA - AK = -AK$ and therefore $A^3 = -AKA = O_n$ and we can conclude as in Solution 3.

Alternatively, since $KA = O_n$, then $A^2 = BA$ and so

$$A^2K^2 = A(AK)K = A(-A^2)K = A(ABA) = A^4 = O_n$$

and $AK^2 = (-A^2)K = ABA$. Then

$$\operatorname{rank}(A^2K) + \operatorname{rank}(AK^2) \leqslant \operatorname{rank}(AK) + \operatorname{rank}(A^2K^2).$$

implies that $2\operatorname{rank}(ABA) \leqslant \operatorname{rank}(AK) \leqslant 1$ which leads to $ABA = O_n$.