Problem 4. Find all bijective functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$x^n f(x) + y^n f(y) \geqslant 2x^n f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$, where n is a fixed positive integer.

Solution 1.

Setting $x = 0$ in (1) gives $y^n f(y) \geqslant 0$, for all $y \in \mathbb{R}$. If n is even, then $f(y) \geqslant 0$ for all y, hence f cannot be surjective and there are no solutions. Therefore, we only have to consider the case where n is odd.

Coming back to $y^n f(y) \geqslant 0$, for all $y \in \mathbb{R}$, since n is odd, this implies that $f(y) \geqslant 0$ for $y > 0$, and $f(y) \leqslant 0$ for $y < 0$. We claim that $f(0) = 0$. Assume for contradiction $f(0) = c \neq 0$. Setting $y = 0$ in (1) gives $x^n f(x) \geqslant 2x^n f(0)$. If $c > 0$, then we get $f(x) > 2c$ for every $x > 0$ and so $\operatorname{Im}(f)$ does not contain any value in $(0, 2c) \setminus \{c\}$. A similar contradiction arises if $c < 0$.

From (1), we have that

$$\frac{f(x)}{f(y)} \geqslant 2 - \left(\frac{y}{x}\right)^n$$

for every $x, y > 0$. Hence, for any fixed positive real number $r < \sqrt[n]{2}$, and any natural number k, we will have

$$\frac{f(x)}{f(rx)} = \frac{f(x)}{f(r^{\frac{1}{k}}x)} \cdot \frac{f(r^{\frac{1}{k}}x)}{f(r^{\frac{2}{k}}x)} \cdot \ldots \cdot \frac{f(r^{\frac{k-1}{k}}x)}{f(rx)} \geqslant \left(2 - r^{n/k}\right)^k.$$

Here, we used the fact that, for $r < \sqrt[n]{2}$, we have $2 - r^{n/k} > 2 - 2^{1/k} \geqslant 0$. Taking the limit as k tends to infinity, we have

$$\frac{f(x)}{f(rx)} \geqslant \lim_{k \to \infty} \left(2 - r^{n/k}\right)^k.$$

To evaluate the limit, we write

$$r^{n/k} = e^{(n \log r)/k} = 1 + \frac{n \log r}{k} + O\left(k^{-2}\right),$$

which gives

$$2 - r^{n/k} = 2 - e^{(n \log r)/k} = 1 - \frac{n \log r}{k} + O\left(k^{-2}\right).$$

Hence,

$$\left(2 - r^{n/k}\right)^k \to e^{-n \log r} = r^{-n},$$

which gives

$$\frac{f(x)}{f(rx)} \geqslant r^{-n}.$$

Similarly, for $r > 1/\sqrt[n]{2}$, we get

$$\frac{f(rx)}{f(x)} \geqslant r^n.$$

Therefore, for $r \in \left(1/\sqrt[n]{2}, \sqrt[n]{2}\right)$, we have $f(rx) = r^n f(x)$. Letting $a = f(1)$, and applying the latter equation repeatedly, we get

$$f(x) = ax^n$$

for every $x > 0$. Similarly, we can show that there exists a constant b such that

$$f(x) = bx^n$$

for every $x < 0$.

In conclusion, if n is even, then there are no solutions. If n is odd, then the solutions are the functions

$$f(x) = \begin{cases} ax^n, & x \geqslant 0, \\ bx^n, & x < 0, \end{cases}$$

where $a, b > 0$ are arbitrary constants. Note that these functions verify (1). Indeed, if $x, y < 0$ or $x, y \geqslant 0$, (1) becomes

$$x^{2n} + y^{2n} \geqslant 2x^n y^n \iff (x^n - y^n)^2 \geqslant 0.$$

If $x < 0$ and $y \geqslant 0$, then (1) becomes $bx^{2n} + ay^{2n} \geqslant 2ax^n y^n$, which is true since $bx^{2n} + ay^{2n} \geqslant 0$ and $2ax^n y^n \leqslant 0$. Finally, if $x \geqslant 0$ and $y < 0$, then (1) becomes $ax^{2n} + by^{2n} \geqslant 2bx^n y^n$, which is true for similar reasons.

Solution 2.

As in Solution 1, we have that n is odd and that the function f preserves signs.

Interchanging x and y in (1) yields

$$x^n f(x) + y^n f(y) \geqslant 2y^n f(x) \tag{2}$$

for all $x, y \in \mathbb{R}$. Adding (1) and (2), we obtain

$$2(x^n f(x) + y^n f(y)) \geqslant 2(x^n f(y) + y^n f(x)),$$

hence

$$(x^n - y^n)(f(x) - f(y)) \geqslant 0 \tag{3}$$

for all $x, y \in \mathbb{R}$. Thus the functions $\varphi(x) = x^n$ and f have the same monotonicity and hence f is increasing. Since f is injective, it is strictly increasing and therefore admits one-sided limits at every point,

$$L^- = \lim_{x \to x_0^-} f(x) \leqslant f(x_0) \leqslant \lim_{x \to x_0^+} f(x) = L^+.$$

Monotonicity together with surjectivity implies $L^- = L^+$, so f is continuous on $\mathbb{R}$.

Write

$$f(x) = \begin{cases} f_1(x), & x > 0, \\ 0, & x = 0, \\ f_2(x), & x < 0, \end{cases}$$

where $f_1 : (0, \infty) \to (0, \infty)$ and $f_2 : (-\infty, 0) \to (-\infty, 0)$ are continuous bijections.

For $0 < y < x$, inequalities (1) and (2) can be rewritten as

$$\frac{f_1(y)}{x^n} \leqslant \frac{f_1(x) - f_1(y)}{x^n - y^n} \leqslant \frac{f_1(x)}{y^n}.$$

Similarly, for $0 < x < y$, it holds that

$$\frac{f_1(x)}{y^n} \leqslant \frac{f_1(x) - f_1(y)}{x^n - y^n} \leqslant \frac{f_1(y)}{x^n}.$$

Letting y tend to x gives

$$\frac{f_1(x)}{x^n} = \lim_{y \to x} \frac{f_1(x) - f_1(y)}{x - y} \cdot \frac{x - y}{x^n - y^n} = \frac{1}{nx^{n-1}} \lim_{y \to x} \frac{f_1(x) - f_1(y)}{x - y},$$

hence f_1 is differentiable and satisfies $f_1'(x) = \dfrac{n}{x} f_1(x)$, for all $x > 0$. Thus

$$\frac{f_1'(x)}{f_1(x)} = \frac{n}{x},$$

which yields $\log(f_1(x)) = n \log x + c$. It follows that $f_1(x) = c_1 x^n$, for all $x > 0$, with $c_1 > 0$ constant ($c_1 = \mathrm{e}^c$). Similarly, $f_2(x) = c_2 x^n$, for all $x < 0$, for some constant $c_2 > 0$.

Therefore, we get the same conclusion as in the first solution.